Chapter 6 Passband Data Transmission

Passband Data Transmission concerns the Transmission of the Digital Data over the real Passband channel.

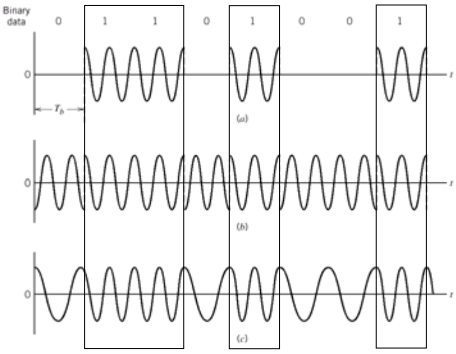
6.1 Introduction – Categories of digital communications (ASK/PSK/FSK)

- Three basic signaling schemes in digital communications

Amplitude-shift keying (ASK)

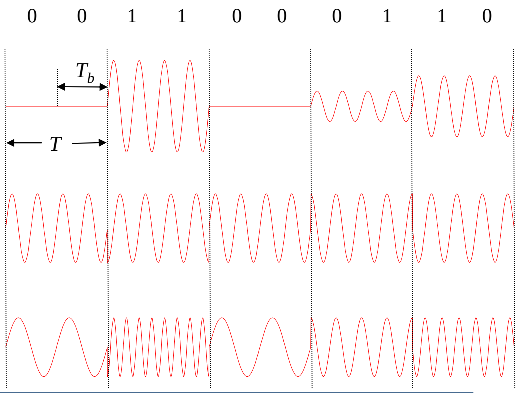
Phase-shift keying (PSK)

Frequency-shift keying (FSK)



6.1 Introduction – Categories of *M*-ary digital communications (ASK/PSK/FSK)

- Three basic signaling schemes in *M*-ary digital communications



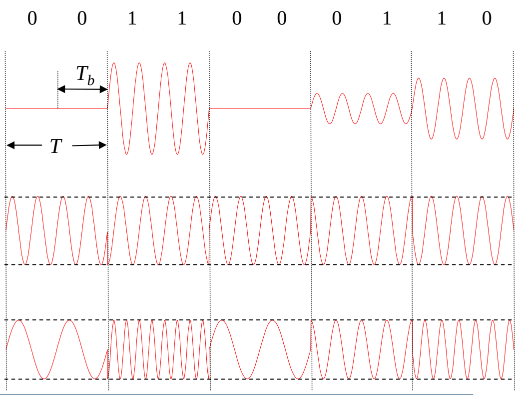
4-ary Amplitude-shift keying (ASK)

4-ary Phase-shift keying (PSK)

4-ary Frequency-shift keying (FSK)

6.1 Introduction – Categories of *M*-ary digital communications (Constant Envelope versus Non-Constant Envelope)

- Constant envelope: A necessity for non-linear channels



Non-constant envelope (ASK)

Constant envelope (PSK)

Constant evnelope (FSK)

6.1 Introduction – Categories of digital communications (Coherent versus Non-Coherent)

- Coherent technique
  - The transmitter and receiver are required to be synchronized in both carrier phase and bit timing.
- Non-Coherent technique
  - The transmitter and receiver are **not** required to be synchronized in both carrier phase and bit timing.

6.1 Introduction – Roadmap

- In this chapter, we will focus on
  - Power : A resource in communication
    - Power Spectra
      - The relation between passband signal and baseband signal is easier to identify in spectra view
  - Bandwidth: Another resource in communication
    - Bandwidth efficiency : The ratio of data rate in bits per second to the effectively utilized bandwidth. (Bits/Second/Hz)
  - Probability of M-ary symbol error (Union bound)

6.1 Introduction – Relation between passband and baseband signals

- Math relation between passband and baseband signals (spectrum view)

$$\tilde{s}(t) = x(t) + jy(t) \quad \text{(Complex) Baseband signal}$$

$$s(t) \quad \text{(Real) Passband signal}$$

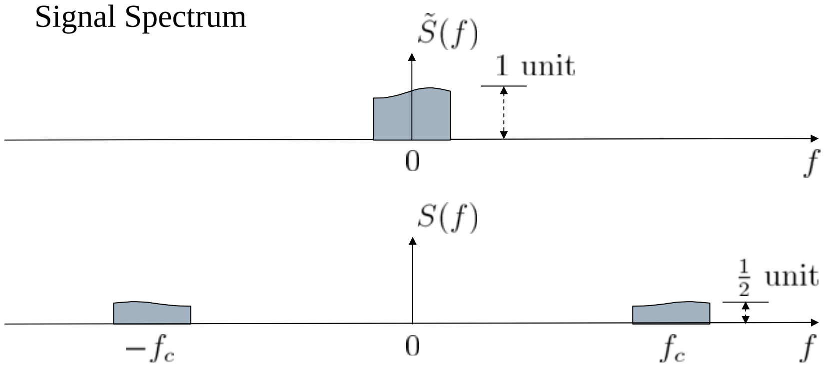
$$\boxed{\begin{aligned} \Rightarrow s(t) &= x(t)\cos(2\pi f_c t) - y(t)\sin(2\pi f_c t) \\ &= \text{Re}\left\{\tilde{s}(t)e^{j2\pi f_c t}\right\} \end{aligned}}$$

6.1 Introduction – Relation between passband and baseband signals

$$\begin{aligned} \boxed{S(f)} &= \int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}dt \\ &= \int_{-\infty}^{\infty} \left\{\text{Re}\left[\tilde{s}(t)e^{j2\pi f_c t}\right]\right\} e^{-j2\pi ft}dt \\ &= \int_{-\infty}^{\infty} \left\{\frac{1}{2}\left[\tilde{s}(t)e^{j2\pi f_c t} + \tilde{s}^*(t)e^{-j2\pi f_c t}\right]\right\} e^{-j2\pi ft}dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \tilde{s}(t)e^{-j2\pi(f-f_c)t}dt + \frac{1}{2}\int_{-\infty}^{\infty} \tilde{s}^*(t)e^{-j2\pi(f+f_c)t}dt \\ &\boxed{= \frac{1}{2}\left[\tilde{S}(f-f_c) + \tilde{S}^*(-f-f_c)\right]} \end{aligned}$$

6.1 Introduction – Relation between passband and baseband signals

Signal Spectrum



6.1 Introduction – Relation between passband and baseband signals

- Math relation between passband and baseband signals (power spectrum view subject to wise-sense stationarity)

$$\tilde{s}(t) = x(t) + jy(t) \quad \text{(Complex) WSS Baseband signal}$$

$$s(t) \quad \text{(Real) WSS Passband signal}$$

$$\boxed{\begin{aligned} \Rightarrow s(t) &= x(t)\cos(2\pi f_c t) - y(t)\sin(2\pi f_c t) \\ &= \mathrm{Re}\left\{\tilde{s}(t)e^{j2\pi f_c t}\right\} \end{aligned}}$$

6.1 Introduction – Relation between passband and baseband signals

- Let

$$\begin{cases} R_{xx}(\tau) = E[x(t+\tau)x(t)] \\ R_{xy}(\tau) = E[x(t+\tau)y(t)] \\ R_{yx}(\tau) = E[y(t+\tau)x(t)] \\ R_{yy}(\tau) = E[y(t+\tau)y(t)] \end{cases}$$

- That $s(t)$ is WSS implies

$$\boxed{R_{ss}(\tau) = E[s(t+\tau)s(t)] \text{ is irrelevant to } t.}$$

6.1 Introduction – Relation between passband and baseband signals

$$\begin{aligned} R_{ss}(\tau) &= E\left[(x(t+\tau)\cos(2\pi f_c(t+\tau)) - y(t+\tau)\sin(2\pi f_c(t+\tau))\right) \\ &\quad (x(t)\cos(2\pi f_c t) - y(t)\sin(2\pi f_c t))] \\ &= R_{xx}(\tau)\cos(2\pi f_c(t+\tau))\cos(2\pi f_c t) + R_{yy}(\tau)\sin(2\pi f_c(t+\tau))\sin(2\pi f_c t) \\ &\quad - R_{xy}(\tau)\cos(2\pi f_c(t+\tau))\sin(2\pi f_c t) - R_{yx}(\tau)\sin(2\pi f_c(t+\tau))\cos(2\pi f_c t) \\ &= R_{xx}(\tau)\frac{\cos(2\pi f_c t) + \cos(2\pi f_c(2t+\tau))}{2} + R_{yy}(\tau)\frac{\cos(2\pi f_c t) - \cos(2\pi f_c(2t+\tau))}{2} \\ &\quad - R_{xy}(\tau)\frac{\sin(2\pi f_c(2t+\tau)) - \sin(2\pi f_c t)}{2} - R_{yx}(\tau)\frac{\sin(2\pi f_c(2t+\tau)) + \sin(2\pi f_c t}{2} \\ &= \frac{1}{2}[R_{xx}(\tau) + R_{yy}(\tau)]\cos(2\pi f_c\tau) + \cancel{\frac{1}{2}[R_{xx}(\tau) - R_{yy}(\tau)]\cos(2\pi f_c(2t+\tau})) \\ &\quad + \frac{1}{2}[R_{xy}(\tau) - R_{yx}(\tau)]\sin(2\pi f_c\tau) - \cancel{\frac{1}{2}[R_{xy}(\tau) + R_{yx}(\tau)]\sin(2\pi f_c(2t+\tau})) \end{aligned}$$

$$\boxed{\text{Then, } R_{xx}(\tau) = R_{yy}(\tau) \text{ and } R_{xy}(\tau) = -R_{yx}(\tau).}$$

6.1 Introduction – Relation between passband signal and baseband signal

$$
\begin{aligned}
R_{\tilde{s}\tilde{s}}(\tau) &= E[(x(t+\tau)+jy(t+\tau))(x(t)+jy(t))^*] \\
&= R_{xx}(\tau)+R_{yy}(\tau)+jR_{yx}(\tau)-jR_{xy}(\tau) \\
&= 2[R_{xx}(\tau)+jR_{yx}(\tau)]
\end{aligned}
$$

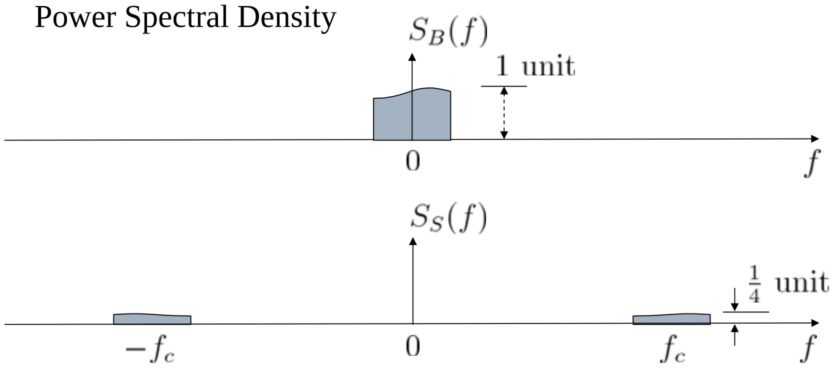
$$
\begin{aligned}
R_{ss}(\tau) &= R_{xx}(\tau)\cos(2\pi f_c\tau)-R_{yx}(\tau)\sin(2\pi f_c\tau) \\
&= \mathrm{Re}\left\{[R_{xx}(\tau)+jR_{yx}(\tau)]\,e^{j2\pi f_c\tau}\right\} \\
&= \frac{1}{2}\mathrm{Re}\left\{R_{\tilde{s}\tilde{s}}(\tau)e^{j2\pi f_c\tau}\right\}
\end{aligned}
$$

6.1 Introduction – Relation between passband signal and baseband signal

$$
\begin{aligned}
\boxed{S_S(f)} &= \int_{-\infty}^{\infty} R_{ss}(\tau)e^{-j2\pi f\tau}d\tau \\
&= \int_{-\infty}^{\infty}\left\{\frac{1}{2}\mathrm{Re}\left[R_{\tilde{s}\tilde{s}}(\tau)e^{j2\pi f_c\tau}\right]\right\}e^{-j2\pi f\tau}d\tau \\
&= \int_{-\infty}^{\infty}\left\{\frac{1}{4}\left[R_{\tilde{s}\tilde{s}}(\tau)e^{j2\pi f_c\tau}+R^*_{\tilde{s}\tilde{s}}(\tau)e^{-j2\pi f_c\tau}\right]\right\}e^{-j2\pi f\tau}d\tau \\
&= \frac{1}{4}\int_{-\infty}^{\infty}R_{\tilde{s}\tilde{s}}(\tau)e^{-j2\pi(f-f_c)\tau}d\tau+\frac{1}{4}\int_{-\infty}^{\infty}R^*_{\tilde{s}\tilde{s}}(\tau)e^{-j2\pi(f+f_c)\tau}d\tau \\
&\boxed{= \frac{1}{4}\left[S_B(f-f_c)+S_B^*(-f-f_c)\right]}
\end{aligned}
$$

6.1 Introduction – Relation between passband signal and baseband signal

Power Spectral Density



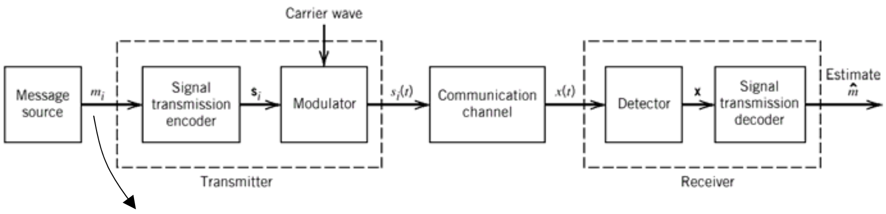
Since $R^*(\tau) = (E[a(t+\tau)a^*(t)])^* = E[a(t)a^*(t+\tau)] = R(-\tau)$, the power spectral density is always real.

6.1 Introduction – Relation between passband signal and baseband signal

- Integration of the Power Spectral Density gives the Power.
  - Integration of the Prabability Density gives the Prabability.

$$
\begin{aligned}
R_{ss}(0) &= \int_{-\infty}^{\infty} S_S(f)df \\
&= \frac{1}{4}\left[\int_{-\infty}^{\infty} S_B(f-f_c)df + \int_{-\infty}^{\infty} S_B(-f-f_c)df\right] \\
&= \frac{1}{4}\left[R_{\tilde{s}\tilde{s}(0)} + R_{\tilde{s}\tilde{s}}(0)\right] = \frac{1}{2}R_{\tilde{s}\tilde{s}}(0)
\end{aligned}
$$

6.2 Passband transmission model – Message source

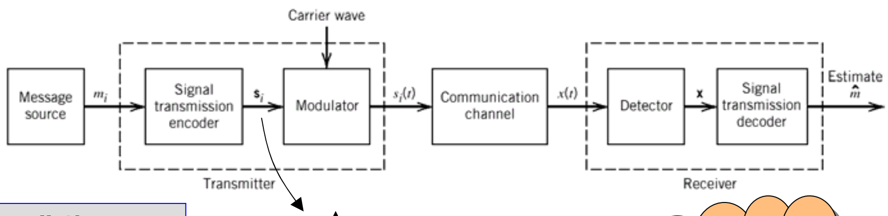


$$m_i \in \{m_1, m_2, \cdots, m_M\}$$

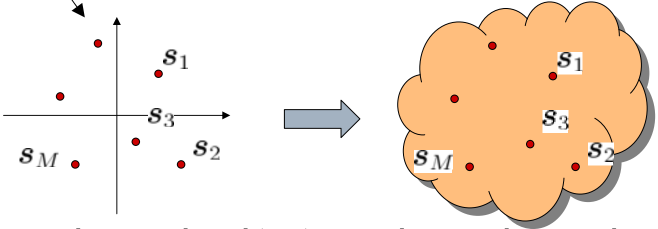
$$\text{prior probability } p_i = P(m_i)$$

$$\text{equal prior } p_i = \frac{1}{M}$$

6.2 Passband transmission model – Signal space analysis

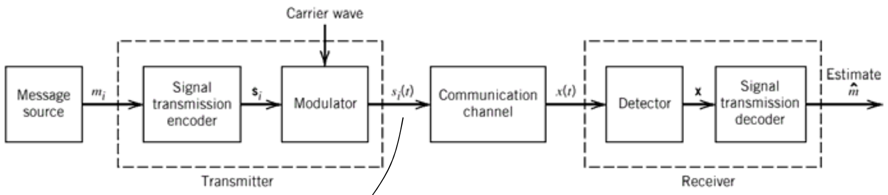


Recall Chapter 5: Signal Space Analysis



2-dimensional vector codeword (N=2) N-dimensional vector codeword

6.2 Passband transmission model – Signal space analysis



Recall Chapter 5: Signal Space Analysis

$$\boldsymbol{s}_i = \begin{bmatrix} s_{i,1} \\ s_{i,2} \\ \vdots \\ s_{i,N} \end{bmatrix} \Rightarrow s_i(t) = \sum_{k=1}^{N} s_{i,k} \cdot \phi_k(t)$$

$\{\phi_k(t)\}_{t=1}^{N}$ is an orthonormal basis

6.2 Passband transmission model – Signal space analysis

- $s_i(t)$ is an (finite) energy signal of duration T.
  - What is an energy signal?

    Define the inner product of two signals $f(t)$ and $g(t)$ as

    $$\langle f(t), g(t)\rangle = \int_0^T f(t)g(t)dt.$$

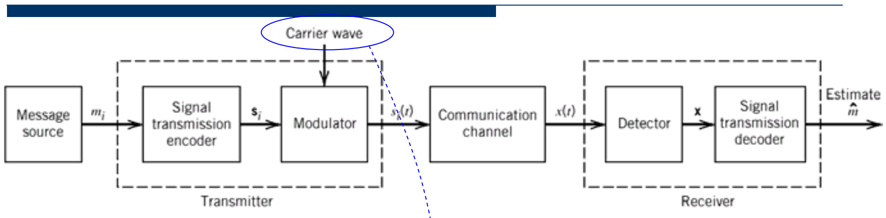
    Then

    $$\begin{aligned} \text{energy of signal } s_i(t) &= \langle s_i(t), s_i(t)\rangle \quad \left(= \|s_i(t)\|^2\right) \\ &= \int_0^T s_i^2(t)dt < 0 \end{aligned}$$

6.2 Passband transmission model – Signal space analysis

$$
\begin{aligned}
\langle s_i(t), s_i(t)\rangle &= \left\langle \sum_{k=1}^{N} s_{i,k}\cdot\phi_k(t), \sum_{\ell=1}^{N} s_{i,\ell}\cdot\phi_\ell(t)\right\rangle \\
&= \sum_{k=1}^{N}\sum_{\ell=1}^{N} \langle s_{i,k}\cdot\phi_k(t), s_{i,\ell}\cdot\phi_\ell(t)\rangle \\
&= \sum_{k=1}^{N}\sum_{\ell=1}^{N} s_{i,k}s_{i,\ell}\langle \phi_k(t), \phi_\ell(t)\rangle \\
&= \sum_{k=1}^{N} s_{i,k}^2
\end{aligned}
$$

6.2 Passband transmission model – Signal space analysis

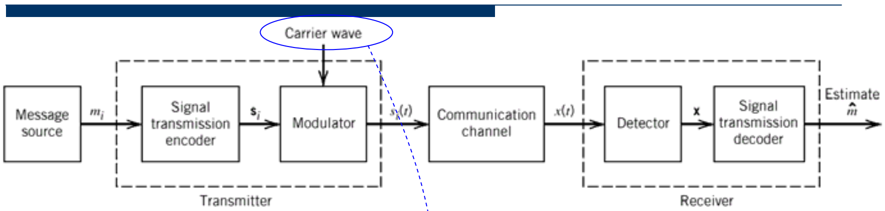


- Example of orthonormal baiss $\{\phi_k(t)\}_{k=1}^{N}$ with $N=2$ for ASK/PSK signals

$$\left\{\sqrt{\frac{2}{T}}\cos(2\pi f_c t), \sqrt{\frac{2}{T}}\sin(2\pi f_c t)\right\}$$

with T being a multiple of $\dfrac{1}{f_c}$

6.2 Passband transmission model – Signal space analysis



- Example of orthonormal baiss $\{\phi_k(t)\}_{k=1}^{N}$ with $N = 2$ for FSK signals

$$\left\{\sqrt{\frac{2}{T}}\cos\left(2\pi\left(f_c - \frac{1}{2T}\right)t\right), \sqrt{\frac{2}{T}}\cos\left(2\pi\left(f_c + \frac{1}{2T}\right)t\right)\right\}$$

with T being a multiple of $\dfrac{1}{f_c}$

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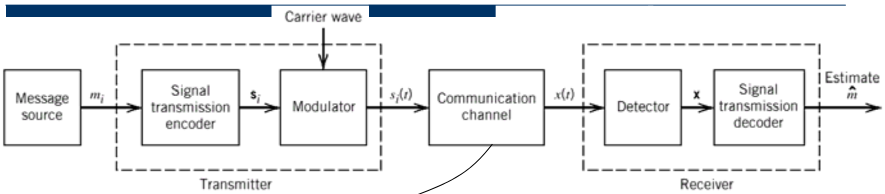
6.2 Passband transmission model – Communication channel



- Communication channel
  - Linear: Principle of superposition

    $s_1(t) \mapsto x_1(t)$ and $s_2(t) \mapsto x_2(t) \Rightarrow as_1(t) + bs_2(t) \mapsto ax_1(t) + bx_2(t)$
  - Sufficient Bandwidth

    No loss of power in $s_i(t)$
  - AWGN

    $x(t) = s_i(t) + n(t)$, where $n(t)$ is a zero-mean white Gaussisn process with two-sided power spectrum density $N_0/2$

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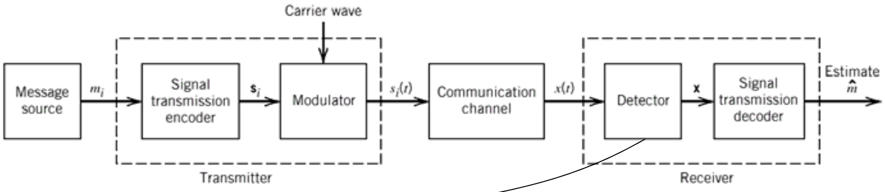
6.2 Passband transmission model – Signal space analysis

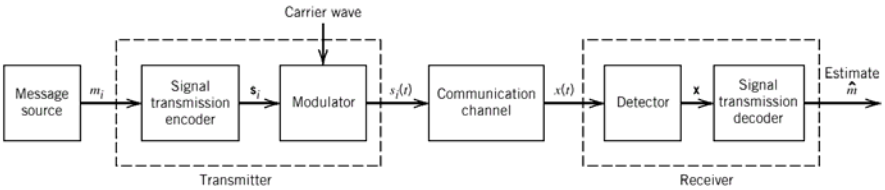


- Detector

$$x(t) = \sum_{k=1}^{N} x_k \cdot \phi_k(t) + \phi'(t) \quad \Rightarrow \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

$$\boxed{\left\langle \phi'(t), \sum_{k=1}^{N} x_k \cdot \phi_k(t) \right\rangle = 0}$$

6.2 Passband transmission model – Decoder

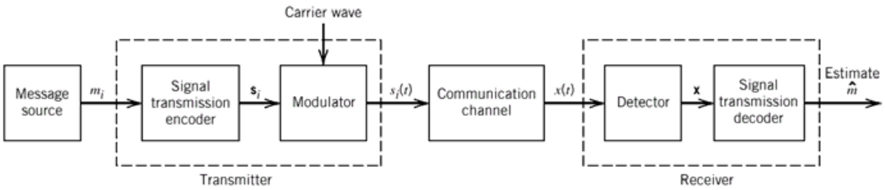


- Signal transmission decoder

$\hat{m}$ is the most probable transmitted message in $\{m_1, m_2, \cdots, m_M\}$ given $\boldsymbol{x}$.

$$\hat{m} = \arg \max_{1 \le i \le M} P(m_i | \boldsymbol{x})$$

6.3 Coherent phase-shift keying

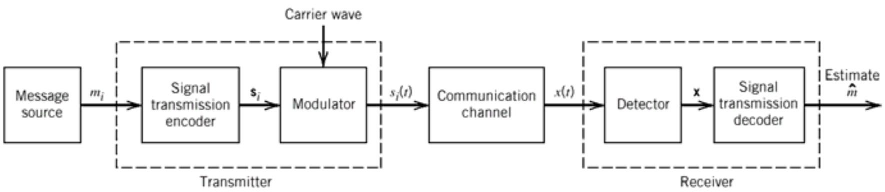


- Binary PSK

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t)$$
$$s_2(t) = \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t + \pi) = -\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t)$$

for $0 \le t < T_b$, where T_b is a multiple of $1/f_c$.

6.3 Coherent phase-shift keying – Antipodal signal



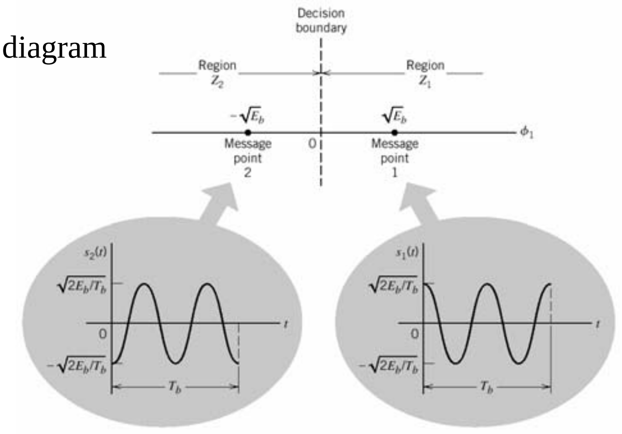
- Vector space analysis of binary PSK
  - Antipodal signal

$$s_1(t) = +\sqrt{E_b}\cdot\phi_1(t) \qquad s_{11} = \langle s_1(t), \phi_1(t)\rangle = +\sqrt{E_b}$$
$$s_2(t) = -\sqrt{E_b}\cdot\phi_1(t) \quad \Rightarrow \quad s_{12} = \langle s_2(t), \phi_1(t)\rangle = -\sqrt{E_b}$$

where $\phi_1(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)$.

6.3 Coherent phase-shift keying – Signal space analysis

Signal-space diagram



6.3 Coherent phase-shift keying – Optimal decision region

- Error probability of binary PSK

$$
\begin{aligned}
& x(t) = s(t) + w(t) \\
\Rightarrow\ & \langle x(t), \phi_1(t)\rangle = \langle s(t), \phi_1(t)\rangle + \langle w(t), \phi_1(t)\rangle \\
\Rightarrow\ & x = \pm\sqrt{E_b} + w \\
\Rightarrow\ & \hat{m} = \arg\max\left\{P\left(x \middle| -\sqrt{E_b}\right), P\left(x \middle| +\sqrt{E_b}\right)\right\} \\
\Rightarrow\ & \hat{m} = \arg\max\left\{\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x+\sqrt{E_b})^2/2\sigma^2}, \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\sqrt{E_b})^2/2\sigma^2}\right\} \\
\Rightarrow\ & x \underset{+\sqrt{E_b}}{\overset{-\sqrt{E_b}}{\lessgtr}} 0
\end{aligned}
$$

$\sigma^2 = N_0/2$ is the variance of w

$\Phi(-x) = \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right)$

Recall:

$$
\begin{aligned}
\sigma^2 = E[w^2] &= \langle w(t), \phi_1(t)\rangle^2 \\
&= E\left[\int_0^{T_b}\int_0^{T_b} w(t)\phi_1(t)\cdot w(s)\phi_1(s)dtds\right] \\
&= \int_0^{T_b}\int_0^{T_b} E[w(t)w(s)]\phi_1(t)\phi_1(s)dtds \\
&= \int_0^{T_b}\int_0^{T_b} \frac{N_0}{2}\delta(t-s)\phi_1(t)\phi_1(s)dtds \\
&= \frac{N_0}{2}\int_0^{T_b}\phi_1(t)\phi_1(t)dt \\
&= \frac{N_0}{2}\langle\phi_1(t),\phi_1(t)\rangle = \frac{N_0}{2}
\end{aligned}
$$

6.3 Coherent phase-shift keying – Error probability

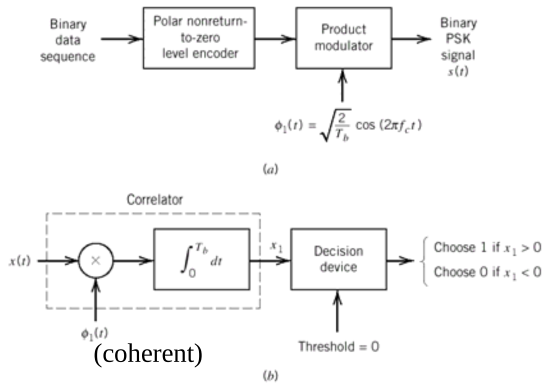
- Error probability of Binary PSK
  - Based on the decision rule $x \underset{+\sqrt{E_b}}{\overset{-\sqrt{E_b}}{\lessgtr}} 0$

$$
\begin{aligned}
P(\text{Error}) &= P\left(-\sqrt{E_b}\text{ transmitted}\right) P\left(x > 0 \middle| -\sqrt{E_b}\text{ transmitted}\right) \\
&\quad + P\left(+\sqrt{E_b}\text{ transmitted}\right) P\left(x < 0 \middle| +\sqrt{E_b}\text{ transmitted}\right) \\
&= \frac{1}{2}\int_0^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x+\sqrt{E_b})^2/2\sigma^2}dx + \frac{1}{2}\int_{-\infty}^{0}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\sqrt{E_b})^2/2\sigma^2}dx \\
&= \int_{-\infty}^{0}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\sqrt{E_b})^2/2\sigma^2}dx = \Phi\left(\frac{0-\sqrt{E_b}}{\sigma}\right) = \Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right)
\end{aligned}
$$

$$\boxed{\Phi(-x) = \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right)}$$

6.3 Coherent phase-shift keying – Block diagram

- Block diagram for PSK transmitter and (coherent) receiver



(coherent)

6.3 Coherent phase-shift keying – Baseband signal

- (Complex) Baseband signal of binary PSK passband signal

$$\begin{aligned} s(t) &= \pm\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t) \\ &= \mathrm{Re}\left\{\left(\pm\sqrt{\frac{2E_b}{T_b}}\right)e^{j2\pi f_c t}\right\} \\ &= \mathrm{Re}\left\{\tilde{s}(t)e^{j2\pi f_c t}\right\} \\ \Rightarrow \tilde{s}(t) &= \pm\sqrt{\frac{2E_b}{T_b}} \text{ for } 0 \le t < T_b \end{aligned}$$

6.3 Coherent phase-shift keying – Sequential baseband signal

- Sequence of complex baseband signals
  - No autocorrelation function of one-shot single random variable.
  - Calculation of the autocorrelation function requires a random process.

$$\tilde{s}(t) = \sum_{k=-\infty}^{\infty} I_k \cdot g(t - kT_b),$$

where $I_k = \pm 1$ with equal probability, and $\{I_k\}_{k=-\infty}^{\infty}$ i.i.d.

$$\text{and } g(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}}, & 0 \le t < T_b \\ 0, & \text{otherwise} \end{cases}$$

6.3 Coherent phase-shift keying – Autocorrelation function

$$\begin{aligned} R_{\tilde{s}\tilde{s}}(t+\tau, t) &= E\left[\left(\sum_{k=-\infty}^{\infty} I_k \cdot g(t+\tau-kT_b)\right)\left(\sum_{\ell=-\infty}^{\infty} I_\ell \cdot g(t-\ell T_b)\right)\right] \\ &= \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} E[I_k I_\ell] g(t+\tau-kT_b) g(t-\ell T_b) \\ &= \sum_{k=-\infty}^{\infty} g(t+\tau-kT_b) g(t-kT_b) \end{aligned}$$

$$
\begin{aligned}
\boxed{\bar{S}_B(f)} &= \int_{-\infty}^{\infty} \bar{R}_{\tilde{s}\tilde{s}}(\tau) e^{-j2\pi f\tau} d\tau \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{T_b} \int_0^{T_b} R_{\tilde{s}\tilde{s}}(t+\tau, t) dt \right) e^{-j2\pi f\tau} d\tau \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{T_b} \int_0^{T_b} \sum_{k=-\infty}^{\infty} g(t+\tau-kT_b) g(t-kT_b) dt \right) e^{-j2\pi f\tau} d\tau \\
&= \frac{1}{T_b} \sum_{k=-\infty}^{\infty} \int_0^{T_b} \left(\int_{-\infty}^{\infty} g(t+\tau-kT_b) e^{-j2\pi f\tau} d\tau \right) g(t-kT_b) dt \\
&= \frac{1}{T_b} G(f) \sum_{k=-\infty}^{\infty} \int_0^{T_b} g(t-kT_b) e^{j2\pi f(t-kT_b)} dt \\
&= \frac{1}{T_b} G(f) \sum_{k=-\infty}^{\infty} \int_{-kT_b}^{(1-k)T_b} g(u) e^{j2\pi f u} du \\
&= \frac{1}{T_b} G(f) \int_{-\infty}^{\infty} g(u) e^{-j2\pi(-f)u} du = \boxed{\frac{1}{T_b} G(f) G(-f) \left(= \frac{1}{T_b} \psi_g(f) \right)}
\end{aligned}
$$

Text denotes $\psi_g(f) \triangleq G(f)G(-f)$, which is called *energy spectra density*.

6.3 Coherent phase-shift keying – Autocorrelation function

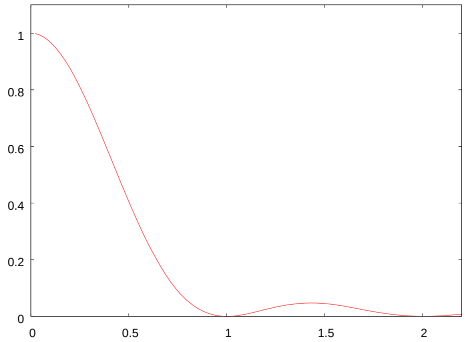
- Power spectrum of binary PSK

$$
G(f) = \int_0^{T_b} \sqrt{\frac{2E_b}{T_b}} e^{-j2\pi ft} dt = \sqrt{\frac{2E_b}{T_b}} \frac{\sin(\pi f T_b)}{\pi f} e^{-j\pi f T_b}
$$

$$
\Rightarrow \boxed{\bar{S}_B(f) = \frac{1}{T_b} G(f) G(-f)} = \frac{2E_b \sin^2(\pi T_b f)}{(\pi T_b f)^2}
$$

6.3 Coherent phase-shift keying – Autocorrelation function

- Power spectrum of binary PSK



6.3 Coherent phase-shift keying – Quadriphase-shift keying

- QPSK

$$s_i(t) = \begin{cases} \sqrt{\dfrac{2E}{T}} \cos\left[2\pi f_c t + (2i-1)\dfrac{\pi}{4}\right], & 0 \le t < T \\ 0, & \text{elsewhere} \end{cases}$$

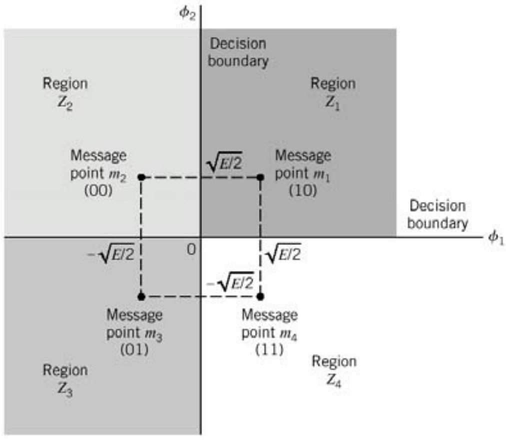
where $i = 1, 2, 3, 4$, f_c is a multiple of $1/T$,
E is the transmitted energy per symbol, and
T is the symbol duration.

- Vector space analysis of QPSK

$$\boldsymbol{s}_i = \begin{bmatrix} \sqrt{E}\cos\left((2i-1)\frac{\pi}{4}\right) \\ -\sqrt{E}\sin\left((2i-1)\frac{\pi}{4}\right) \end{bmatrix} \text{ with } \begin{cases} \phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t) \\ \phi_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_c t) \end{cases}$$

6.3 Coherent phase-shift keying – Quadriphase-shift keying

- Two-dimensional signal space diagram of QPSK



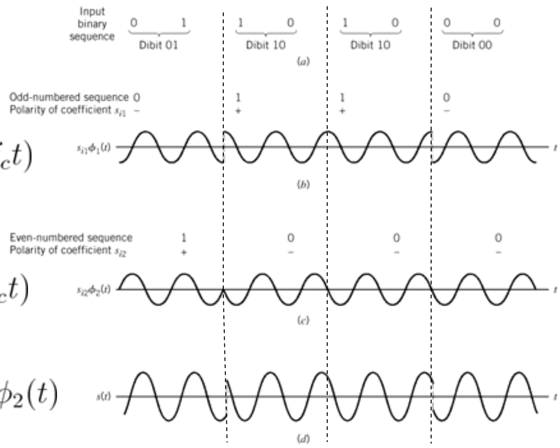
6.3 Coherent phase-shift keying – Quadriphase-shift keying

- Example 6.1

$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)$$

$$\phi_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_c t)$$

$$s(t) = s_{i1}\phi_1(t) + s_{i2}\phi_2(t)$$



- Error probability of QPSK

$$x(t) = s(t) + w(t)$$

$$\Rightarrow \begin{cases} \langle x(t), \phi_1(t)\rangle = \langle s(t), \phi_1(t)\rangle + \langle w(t), \phi_1(t)\rangle \\ \langle x(t), \phi_2(t)\rangle = \langle s(t), \phi_2(t)\rangle + \langle w(t), \phi_2(t)\rangle \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = \pm\sqrt{\frac{E}{2}} + w_1 \\ x_2 = \pm\sqrt{\frac{E}{2}} + w_2 \end{cases}$$

$$\Rightarrow \quad \hat{m} = \arg\max \left\{ P\left(x_1, x_2 \,\middle|\, \pm\sqrt{E/2}, \pm\sqrt{E/2}\right)\right\}$$

$$\Rightarrow \quad \hat{m} = \arg\max \left\{ \frac{1}{2\pi\sigma^2} e^{-[(x_1 \mp \sqrt{E/2})^2 + (x_2 \mp \sqrt{E/2})^2]/(2\sigma^2)} \right\}$$

$$\Rightarrow \quad x_1 \underset{+\sqrt{E/2}}{\overset{-\sqrt{E/2}}{\lessgtr}} 0 \text{ and } x_2 \underset{+\sqrt{E/2}}{\overset{-\sqrt{E/2}}{\lessgtr}} 0$$

$$\boxed{\sigma^2 = N_0/2}$$

6.3 Coherent phase-shift keying – Error probability of QPSK

- Following the same derivation as that in slide 5-32

$$\Pr(s_1 \text{ error}) = \Phi\left(-\sqrt{2\frac{E/2}{N_0}}\right) = \Phi\left(-\sqrt{\frac{E}{N_0}}\right)$$

$$\Pr(s_2 \text{ error}) = \Phi\left(-\sqrt{2\frac{E/2}{N_0}}\right) = \Phi\left(-\sqrt{\frac{E}{N_0}}\right)$$

Since $E = 2E_b$,

$$\text{Bit Error Rate} = \Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right)$$

if s_1 and s_2 respectively decide one information bit as shown on 6-41.

$$\boxed{\Phi(-x) = \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right)}$$

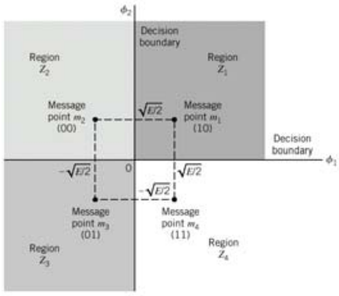
6.3 Coherent phase-shift keying – Error probability of QPSK

- Symbol error rate of QPSK

$$
\begin{aligned}
& \Pr(\text{Symbol Error}) \\
&= 1 - \Pr(\text{Symbol Correct}) \\
&= 1 - \Pr(s_1 \text{ Correct}) \Pr(s_2 \text{ Correct}) \\
& \quad (\text{Because the noise affecting the two decisions are independent}) \\
&= 1 - \left[1 - \Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right)\right]^2 \\
&= 2\Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right) - \Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right)^2
\end{aligned}
$$

6.3 Coherent phase-shift keying – Error probability of QPSK

- Alternative approach to derive the symbol error rate of QPSK



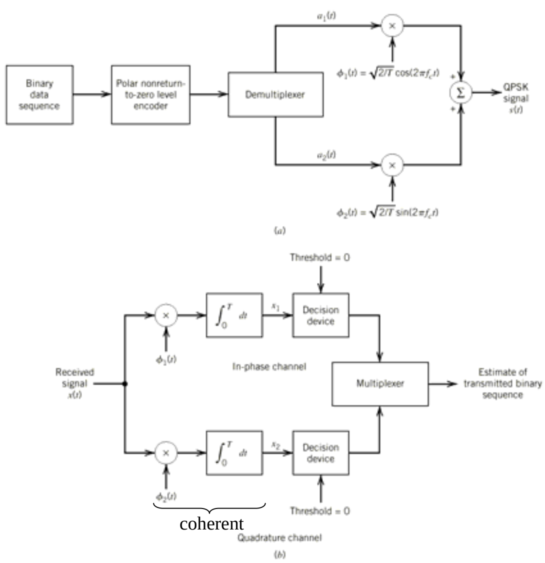
$$
\begin{aligned}
& \Pr(\text{Symbol Error} | s_1 = \sqrt{E/2}, s_2 = -\sqrt{E/2}) \\
&= \int_{\substack{\text{shaded}\\\text{area}}} p(x_1, x_2 | \sqrt{E/2}, -\sqrt{E/2}) dx_1 dx_2 \\
&= \int_{\substack{\text{shaded}\\\text{area}}} \frac{1}{2\pi\sigma^2} e^{-[(x_1-\sqrt{E/2})^2+(x_2+\sqrt{E/2})^2]/2\sigma^4} dx_1 dx_2 \\
&= \cdots (\text{omit}) \\
&= 2\Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right) - \Phi\left(-\sqrt{2\frac{E_b}{N_0}}\right)^2
\end{aligned}
$$

6.3 Coherent phase-shift keying – Summary

- Partial summary
  - QPSK (with Gray code mapping) and BPSK have the same BER under the same E_b/N_0.
  - QPSK however doubles the transmission bit rate per second (or uses half the bandwidth under the same bit rate) by introducing another quadrature.
  - In its implementation, QPSK is more complex since it involves two quadratures.

6.3 Coherent phase-shift keying – QPSK diagram

- Block diagram of QPSK transmitter and receiver



6.3 Coherent phase-shift keying – Sequential baseband signal

- Sequence of complex baseband signals
  - No autocorrelation function of one-shot (namely, single) random variable.
  - Calculation of the autocorrelation function requires a random process.

$$\tilde{s}(t) = \sum_{k=-\infty}^{\infty} e^{j[(\pi/2)I_k+\pi/4]} g(t-kT),$$

where $I_k = 0,1,2,3$ with equal prob., and $\{I_k\}_{k=-\infty}^{\infty}$ i.i.d.

$$\text{and } g(t) = \begin{cases} \sqrt{\frac{2E}{T}}, & 0 \le t < T = 2T_b \\ 0, & \text{otherwise} \end{cases}$$

6.3 Coherent phase-shift keying – Sequential baseband signal

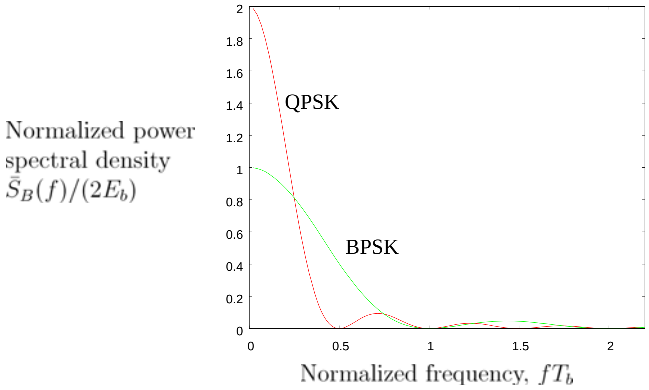
$$\begin{aligned} R_{\tilde{s}\tilde{s}}&(t+\tau,t) \\ &= E\left[\left(\sum_{k=-\infty}^{\infty} e^{j[(\pi/2)I_k+\pi/4]} g(t+\tau-kT)\right)\left(\sum_{\ell=-\infty}^{\infty} e^{j[(\pi/2)I_\ell+\pi/4]} g(t-\ell T)\right)^*\right] \\ &= \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} g(t+\tau-kT)g(t-\ell T)E\left[e^{j(\pi/2)(I_k-I_\ell)}\right] \\ &= \sum_{k=-\infty}^{\infty} g(t+\tau-kT)g(t-kT) \end{aligned}$$

$\boxed{E[e^{j(\pi/2)I_k}] = 0 \text{ for uniform prior}}$

$$\Rightarrow \bar{S}_B(f) = \frac{1}{T}G(f)G(-f) = \frac{2E\sin^2(\pi T f)}{(\pi T f)^2} = \frac{4E_b \sin^2(2\pi T_b f)}{(2\pi T_b f)^2}$$

6.3 Coherent phase-shift keying – Autocorrelation function

- Power spectrum of BPSK and QPSK under the same E_b and T_b



$$s(t) = \Re\left\{e^{j[(\pi/2)I_k+\pi/4]}e^{j2\pi f_c t}g(t)\right\} \text{ for } I_k = 0(++),\underbrace{1(-+)}\underbrace{,2(--)}\underbrace{,3(+-)}$$

Single sign change = 90 degree shift
Double sign change = 180 degree shift

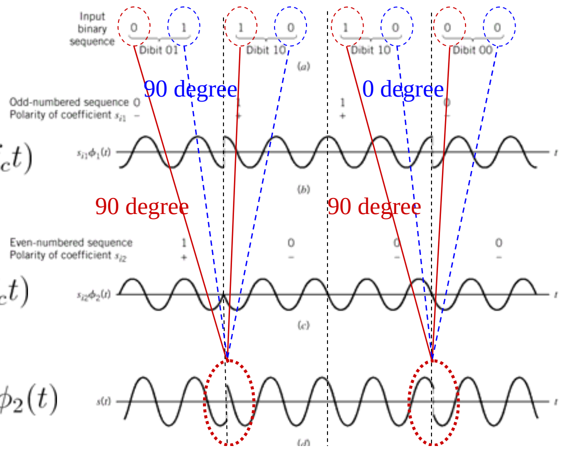
6.3 Coherent phase-shift keying – Offset QPSK

- Example 6.1:
  - QPSK

$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)$$

$$\phi_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_c t)$$

$$s(t) = s_{i1}\phi_1(t) + s_{i2}\phi_2(t)$$



90+90=180 degree phase shift

90+0=90 degree phase shift

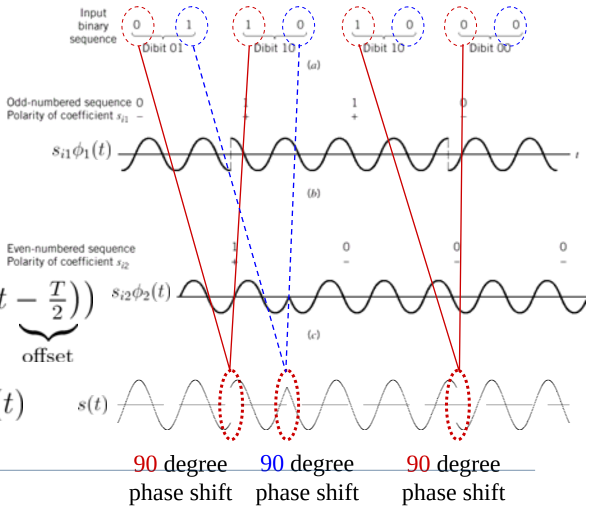
6.3 Coherent phase-shift keying – Offset QPSK

- Example 6.1:
  - Offset QPSK

$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)$$

$$\phi_2(t) = \sqrt{\frac{2}{T}}\sin\left(2\pi f_c \underbrace{\left(t-\frac{T}{2}\right)}_{\text{offset}}\right)$$

$$s(t) = s_{i1}\phi_1(t) + s_{i2}\phi_2(t)$$



6.3 Coherent phase-shift keying – Offset QPSK

- Offset QPSK
  - By "offseting" the quadrature component by half a symbol interval with respect to the in-phase component, Offset QPSK limits the amplitude fluctuation to 90 degree.
  - The 90 degree phase transition in OQPSK occurs twice as frequently encountered in QPSK.
    - Personal comment: One 180 degree phase transition in QPSK becomes two 90 degree phase transitions in OQPSK. Hence, "twice" is an over-estimate.
  - Under AWGN and coherent receiver, the error rate of OQPSK is exactly the same as that of QPSK.

6.3 Coherent phase-shift keying – π/4-shifted DQPSK

- π/4-shifted DQPSK
  - The input dibit does not determine the absolute phase, but the phase change.

$$\text{in-phase} \Rightarrow \cos(\theta_k) = \cos(\theta_{k-1} + \Delta\theta_k)$$

$$\text{quadrature} \Rightarrow \sin(\theta_k) = \sin(\theta_{k-1} + \Delta\theta_k)$$

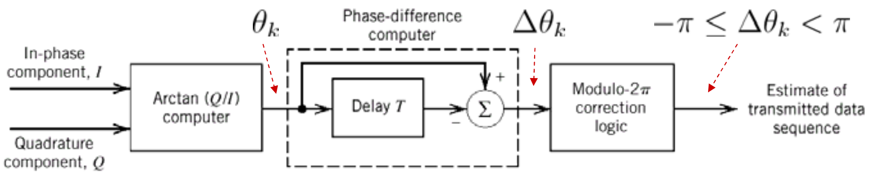
$$\Delta\theta_k = \begin{cases} +\pi/4, & 00 \\ +3\pi/4, & 01 \\ -3\pi/4, & 11 \\ -\pi/4, & 10 \end{cases}$$

6.3 Coherent phase-shift keying – π/4-shifted DQPSK

- π/4-shifted DQPSK
  - The phase transition is restricted to either 45 or 135 degree.
    - No 0 degree phase transition occurs now!
  - Noncoherent receiver is feasible.

6.3 Coherent phase-shift keying – Detection of π/4-shifted DQPSK

- Noncoherent receiver
  - Differential detector



$$\phi_1(t) = \sqrt{\tfrac{2}{T}}\cos(2\pi f_c t) \Rightarrow I = \langle x(t), \phi_1(t)\rangle$$

$$\phi_2(t) = \sqrt{\tfrac{2}{T}}\sin(2\pi f_c t) \Rightarrow Q = \langle x(t), \phi_2(t)\rangle$$

6.3 Coherent phase-shift keying – *M*-ary PSK

- *M*-ary PSK

$$s_i(t) = \begin{cases} \sqrt{\dfrac{2E}{T}}\cos\left[2\pi f_c t + (i-1)\dfrac{2\pi}{M}\right], & 0 \leq t < T \\ 0, & \text{elsewhere} \end{cases}$$

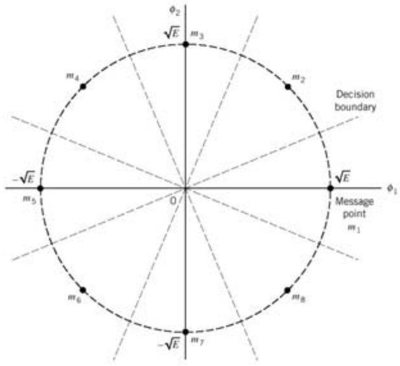
where $i = 1, 2, \ldots, M$, f_c is a multiple of $1/T$,
E is the transmitted energy per symbol, and
T is the symbol duration.

- Vector space analysis of *M*-ary PSK

$$\boldsymbol{s}_i = \begin{bmatrix} \sqrt{E}\cos\left((i-1)\frac{2\pi}{M}\right) \\ -\sqrt{E}\sin\left((i-1)\frac{2\pi}{M}\right) \end{bmatrix} \text{ with } \begin{cases} \phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t) \\ \phi_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_c t) \end{cases}$$

6.3 Coherent phase-shift keying – *M*-ary PSK

- Example – 8PSK (Octaphase-shift keying)



6.3 Coherent phase-shift keying – Union bound of *M*-ary PSK error

$$
\begin{aligned}
P_{e,\text{symbol}} &= \sum_{i=1}^{M} \Pr(m_i \text{ transmitted}) \Pr(\text{decision} \neq m_i | m_i \text{ transmitted}) \\
&= \sum_{i=1}^{M} \frac{1}{M} \Pr\left(\begin{array}{l} \text{decision} = m_1 \\ \text{ or } \cdots \\ \text{ or decision} = m_{i-1} \\ \text{ or decision} = m_{i+1} \\ \text{ or } \cdots \\ \text{ or decision} = m_M \end{array} \middle| \, m_i \text{ transmitted} \right) \\
&\leq \frac{1}{M} \sum_{i=1}^{M} \sum_{\ell=1,\ell\neq i}^{M} \Pr\left(\text{decision} = m_\ell | m_i \text{ transmitted}\right)
\end{aligned}
$$

6.3 Coherent phase-shift keying – Union bound of *M*-ary PSK error

- If the signal constellation is symmetric in the sense that

$$\sum_{\ell=1,\ell\neq i}^{M} \Pr\left(\text{decision} = m_\ell | m_i \text{ transmitted}\right) = \text{constant in } i$$

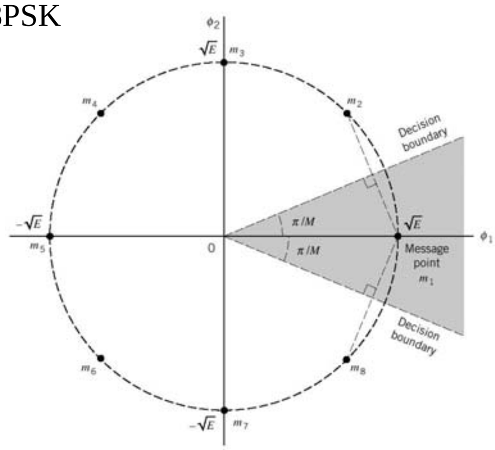
then

$$\begin{aligned} P_{e,\text{symbol}} &\leq \sum_{\ell=1,\ell\neq i}^{M} \Pr\left(\text{decision} = m_\ell | m_i \text{ transmitted}\right) \\ &= \sum_{\ell=1,\ell\neq i}^{M} \Phi\left(-\frac{d_{\ell,i}}{\sqrt{2N_0}}\right) \quad \text{under AWGN (See (5.89))} \end{aligned}$$

$$\boxed{\Phi(-x) = \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right)}$$

6.3 Coherent phase-shift keying – Union bound of *M*-ary PSK error

- Example – 8PSK



6.3 Coherent phase-shift keying – Union bound of *M*-ary PSK error

$$
\begin{aligned}
P_{e,\text{symbol}} &\le \sum_{\ell=1,\ell\neq 1}^{8} \Phi\left(-\frac{d_{\ell,1}}{\sqrt{2N_0}}\right) \quad \text{under AWGN (See (5.89))} \\
&= \Phi\left(-\frac{d_{2,1}}{\sqrt{2N_0}}\right) + \Phi\left(-\frac{d_{3,1}}{\sqrt{2N_0}}\right) + \Phi\left(-\frac{d_{4,1}}{\sqrt{2N_0}}\right) + \Phi\left(-\frac{d_{5,1}}{\sqrt{2N_0}}\right) \\
&+ \Phi\left(-\frac{d_{8,1}}{\sqrt{2N_0}}\right) + \Phi\left(-\frac{d_{7,1}}{\sqrt{2N_0}}\right) + \Phi\left(-\frac{d_{6,1}}{\sqrt{2N_0}}\right) \\
&= 2\Phi\left(-\frac{d_{2,1}}{\sqrt{2N_0}}\right) + 2\Phi\left(-\frac{d_{3,1}}{\sqrt{2N_0}}\right) + 2\Phi\left(-\frac{d_{4,1}}{\sqrt{2N_0}}\right) + \Phi\left(-\frac{d_{5,1}}{\sqrt{2N_0}}\right) \\
&\approx 2\Phi\left(-\frac{d_{2,1}}{\sqrt{2N_0}}\right)
\end{aligned}
$$

$$\boxed{d_{2,1} = 2\sqrt{E}\sin\left(\frac{\pi}{M}\right)}$$

This is the lower bound of the upper bound.
So it is not really an upper bound!

6.3 Coherent phase-shift keying – Power spectra of *M*-ary PSK signals

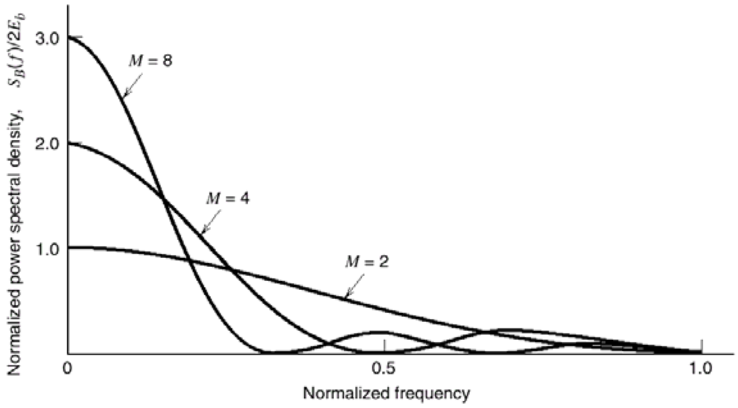
- Same as slide **Chapter 6-50**

$$\bar{S}_B(f) = \frac{2E\sin^2(\pi T f)}{(\pi T f)^2}$$

With $T = T_b \log_2(M)$ and $E = E_b \log_2(M)$

$$\bar{S}_B(f) = \frac{2E_b \log_2(M)\sin^2(\pi T_b \log_2(M) f)}{(\pi T_b \log_2(M) f)^2}$$

6.3 Coherent phase-shift keying – Power spectra of *M*-ary PSK signals



6.3 Coherent phase-shift keying – Bandwidth efficiency

- Bandwidth efficiency of *M*-ary PSK signals
  - Null-to-null bandwidth

$$B = \frac{2}{T} = \frac{2}{T_b \log_2(M)} = \frac{2R_b}{\log_2(M)}$$

  - Bandwidth efficiency

$$\rho \text{ (bits/s/Hz)} = \frac{R_b}{B} = \frac{1}{2}\log_2(M)$$

$$\boxed{P_{e,\text{symbol}} \approx 2\Phi\left(-\sin\left(\frac{\pi}{M}\right)\sqrt{\frac{2E_b \log_2(M)}{N_0}}\right)}$$

6.3 Coherent phase-shift keying – Bandwidth efficiency

- Final note
  - There is a trade-off between symbol error rate and bandwidth efficiency for *M*-ary PSK signals.

6.4 Hybrid amplitude/phase modulation schemes

- For *M*-ary PSK signals, the in-phase and quadrature components are "dependent".
  - How about making them "independent" (to increase the data rate)?
  - Answer: *M*-ary quadrature amplitude modulation (QAM)

$$s_k(t) = a_k\sqrt{E_0}\phi_1(t) + b_k\sqrt{E_0}\phi_2(t)$$

$$\boldsymbol{s}_k = \begin{bmatrix} a_k\sqrt{E_0} \\ b_k\sqrt{E_0} \end{bmatrix} \text{ with } \begin{cases} \phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t) \\ \phi_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_c t) \end{cases}$$

6.4 Hybrid amplitude/phase modulation schemes

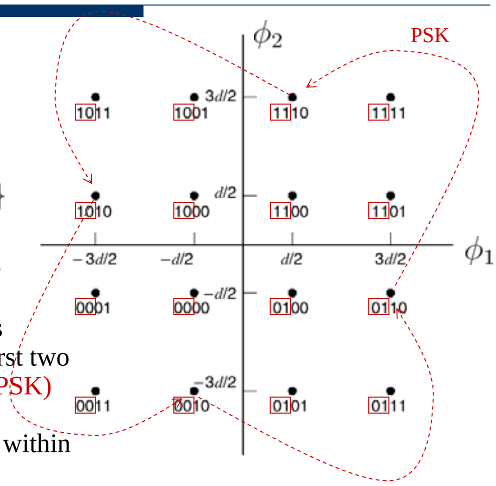
- Square constellation

$$\sqrt{E_0} = \frac{d}{2}$$

$$a_k \in \{-3, -1, +1, +3\}$$

$$b_k \in \{-3, -1, +1, +3\}$$

- Gray-encoded quadbits
  - Gray-encode the first two bits by quadrants (PSK)
  - Gray-encode the remaining two bits within quadrants (ASK)

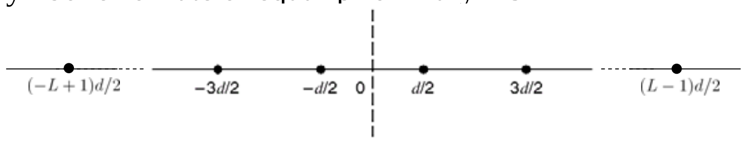


6.4 Hybrid amplitude/phase modulation schemes

- Symbol error rate of square QAM

$$\begin{aligned} P_{e,M-\text{QAM}} &= 1 - P_{c,M-\text{QAM}} \\ &= 1 - (1 - P_{e,\sqrt{M}-\text{ASK}})^2 \\ &= 2P_{e,\sqrt{M}-\text{ASK}} - P^2_{e,\sqrt{M}-\text{ASK}} \\ &\approx 2P_{e,\sqrt{M}-\text{ASK}} \end{aligned}$$

- Symbol error rate of equal-prior L-ary ASK



- Symbol error rate of equal-prior L-ary ASK

$$\begin{aligned} P_{e,L\text{-ASK}} &= \frac{1}{L}\Pr\left(x-(-L+1) > \frac{d}{2}\,\middle|\, s=-L+1\right) \\ &\quad + \sum_{k=2}^{L-1}\frac{1}{L}\Pr\left(|x-(-L+2k-1)| > \frac{d}{2}\,\middle|\, s=-L+2k-1\right) \\ &\quad + \frac{1}{L}\Pr\left(x-(L-1) < -\frac{d}{2}\,\middle|\, s=L-1\right) \\ &= \frac{1}{L}\left[1-\Phi\left(\frac{d/2}{\sqrt{N_0/2}}\right)\right] + \sum_{k=2}^{L-1}\frac{1}{L}2\Phi\left(-\frac{d/2}{\sqrt{N_0/2}}\right) + \frac{1}{L}\Phi\left(-\frac{d/2}{\sqrt{N_0/2}}\right) \\ &= 2\frac{L-1}{L}\Phi\left(-\frac{d/2}{\sqrt{N_0/2}}\right) \\ &= 2\left(1-\frac{1}{L}\right)\Phi\left(-\sqrt{\frac{2E_0}{N_0}}\right) \end{aligned}$$

6.4 Hybrid amplitude/phase modulation schemes

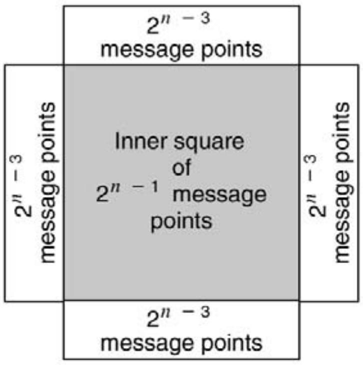
- Average transmitted energy of M-ary QAM

$$\begin{aligned} E_{av} &= \sum_{k=1}^{M}\frac{1}{M}\|\boldsymbol{s}_k\|^2 \\ &= \sum_{k=1}^{M}\frac{1}{M}E_0(a_k^2+b_k^2) \\ &= \frac{E_0}{M}\sum_{k=1}^{L}\sum_{k'=1}^{L}[(-L+2k-1)^2+(-L+2k'-1)^2] \qquad \boxed{L=\sqrt{M}} \\ &= \frac{2}{3}(M-1)E_0 \end{aligned}$$

$$P_{e,M\text{-QAM}} \approx 4\left(1-\frac{1}{\sqrt{M}}\right)\Phi\left(-\sqrt{\frac{3E_{av}}{N_0(M-1)}}\right)$$

6.4 Hybrid amplitude/phase modulation schemes

- Square constellation QAM
  - M is usually even power of two.
  - For example, $M = 2^2, 2^4, 2^6, 2^8, \ldots$ (in such case $L = 2, 2^2, 2^3, 2^4, \ldots$)
- Question: How about $M = 2^3, 2^5, 2^7, \ldots$
  - Asnwer: Cross constellation QAM



$$2^n = 2^{n-1} + 4 \times 2^{n-3}$$

6.4 Hybrid amplitude/phase modulation schemes

- Symbol error rate of cross-constellation QAM

$$P_{e,M-\text{cross}-\text{QAM}} \approx 4\left(1-\frac{1}{\sqrt{2M}}\right)\Phi\left(-\sqrt{\frac{2E_0}{N_0}}\right)$$

$$\boxed{P_{e,M-\text{square}-\text{QAM}} \approx 4\left(1-\frac{1}{\sqrt{M}}\right)\Phi\left(-\sqrt{\frac{2E_0}{N_0}}\right)}$$

6.4 Hybrid amplitude/phase modulation schemes – Carrierless amplitude/phase modulation (CAP)

- QAM can be viewed as one of the family members in carrierless amplitude/phase modulation (CAP)

$$\begin{aligned} s_k(t) &= a_k\sqrt{E_0}\phi_1(t) - b_k\sqrt{E_0}\phi_2(t) \\ &= a_k g(t-kT)\cos(2\pi f_c t) - b_k g(t-kT)\sin(2\pi f_c t) \end{aligned}$$

$$\text{where } g(t) = \begin{cases} \sqrt{\frac{2E_0}{T}}, & 0 \le t < T; \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \begin{cases} \phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t) \\ \phi_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_c t) \end{cases}$$

- Re-express QAM into CAP form

$$\begin{aligned} s(t) &= \sum_{k=-\infty}^{\infty} s_k(t) \\ &= \sum_{k=-\infty}^{\infty} \left(a_k g(t-kT)\cos(2\pi f_c t) - b_k g(t-kT)\sin(2\pi f_c t)\right) \\ &= \mathrm{Re}\left\{\sum_{k=-\infty}^{\infty} (a_k + jb_k) g(t-kT) e^{j2\pi f_c t}\right\} \\ &= \mathrm{Re}\left\{\sum_{k=-\infty}^{\infty} \left((a_k + jb_k) e^{j2\pi f_c kT}\right) \cdot \left(g(t-kT) e^{j2\pi f_c (t-kT)}\right)\right\} \\ &= \mathrm{Re}\left\{\sum_{k=-\infty}^{\infty} \tilde{A}_k \cdot g_+(t-kT)\right\} \end{aligned}$$

Carrierless since no carrier f_c appears in this formula.

where $\tilde{A}_k = (a_k + jb_k)e^{j2\pi f_c kT}$ and $g_+(t) = g(t)e^{j2\pi f_c t}$.

6.4 Hybrid amplitude/phase modulation schemes – Carrierless amplitude/phase modulation (CAP)

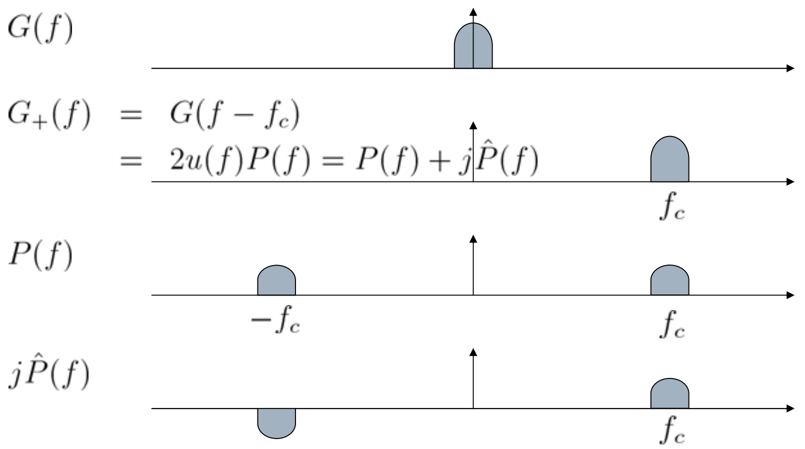
- Properties of the passband in-phase and quadrature pulses in CAP

$$g_+(t) = g(t)\cos(2\pi f_c t) + jg(t)\sin(2\pi f_c t) = p(t) + j\hat{p}(t)$$

$$\text{where } \begin{cases} p(t) = g(t)\cos(2\pi f_c t) \\ \hat{p}(t) = g(t)\sin(2\pi f_c t) \end{cases}$$

  - **Property 0:** $\hat{p}(t)$ is the Hilbert transform of $p(t)$.

6.4 Hybrid amplitude/phase modulation schemes – Carrierless amplitude/phase modulation (CAP)



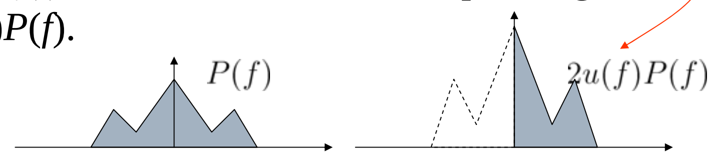
A2.3 Hilbert transform

- Let $P(f)$ be the spectrum of a real function $p(t)$.
  - By convention, denote by $u(f)$ the unit step function, i.e.,

$$u(f) = \begin{cases} 1, & f > 0 \\ 1/2, & f = 0 \\ 0, & f < 0 \end{cases}$$

Multiply by 2 to unchange the area.

- Put $g_+(t)$ to be the function corresponding to $2u(f)P(f)$.



A2.3 Hilbert transform

- How to obtain $g_+(t)$?
- Answer: *Hilbert Transformer*.

Proof: Observe that

$$2u(f) = 1 + \mathrm{sgn}(f), \text{ where } \mathrm{sgn}(f) = \begin{cases} 1, & f > 0 \\ 0, & f = 0 \\ -1, & f < 0 \end{cases}$$

Then by the next slide, we learn that

$$2u(f) \overset{\text{Inverse\_Fourier}}{=} \delta(t) + j\frac{1}{\pi t} \cdot \mathbf{1}\{t \neq 0\}$$

By extended Fourier transform,

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{sgn}(f)e^{-a|f|+j2\pi ft}df &= \int_{0}^{\infty} e^{-a|f|+j2\pi ft}df - \int_{-\infty}^{0} e^{-a|f|+j2\pi ft}df \\
&= \int_{0}^{\infty} e^{-(a-j2\pi t)f}df - \int_{0}^{\infty} e^{-(a+2\pi t)f}df \\
&= \frac{1}{a-j2\pi t} - \frac{1}{a+j2\pi t} \\
&= \frac{j4\pi t}{a^2+4\pi^2t^2}
\end{aligned}$$

$$\mathrm{sgn}(f) \overset{\text{Inverse\_Fourier}}{=} \lim_{a\downarrow 0} j\frac{4\pi t}{a^2+4\pi t^2} = \begin{cases} \frac{j}{\pi t}, & t \neq 0 \\ 0, & t = 0 \end{cases}$$

$$2u(f) = 1 + \mathrm{sgn}(f) \overset{\text{Inverse\_Fourier}}{=} \delta(t) + \frac{j}{\pi t}\cdot \mathbf{1}\{t \neq 0\}$$

A2.3 Hilbert transform

$$\begin{aligned}
g_+(t) &= \mathrm{Fourier}^{-1}\{2u(f)P(f)\} \\
&= \mathrm{Fourier}^{-1}\{2u(f)\} * \mathrm{Fourier}^{-1}\{P(f)\} \\
&= \left(\delta(t) + j\frac{1}{\pi t}\mathbf{1}\{t \neq 0\}\right) * p(t) \\
&= p(t) + j\frac{1}{\pi t}\cdot \mathbf{1}\{t \neq 0\} * p(t) \\
&= p(t) + j\hat{p}(t),
\end{aligned}$$

where $\hat{p}(t) = \int_{-\infty}^{\infty} p(\tau)\frac{1}{\pi(t-\tau)}d\tau$ is the Hilbert transform of $p(t)$.

A2.3 Hilbert transform

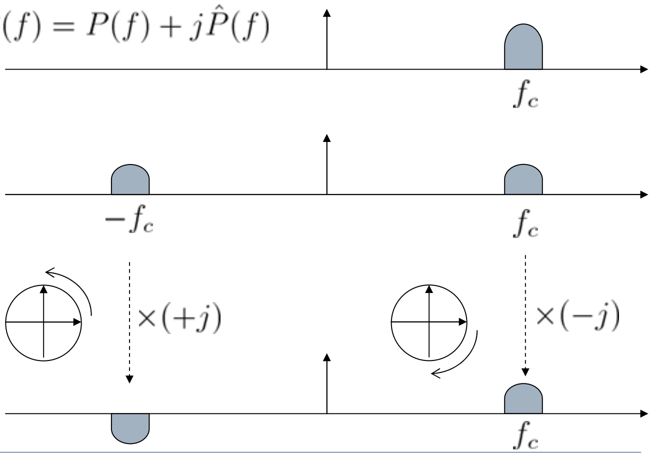
$$p(t) \longrightarrow \boxed{h(\tau)=\frac{1}{\pi\tau}} \longrightarrow \hat{p}(t)$$

$$h(\tau)=\frac{1}{\pi\tau} \Rightarrow H(f)=-j\,\mathrm{sgn}(f),\ \text{where } \mathrm{sgn}(f)=\begin{cases} +1, & f>0 \\ 0, & f=0 \\ -1, & f<0 \end{cases}$$

$$\Rightarrow \hat{P}(f)=(-j\,\mathrm{sgn}(f))\cdot P(f)=\begin{cases} |P(f)|\exp\{j[\angle P(f)-\pi/2]\}, & f>0 \\ 0, & f=0 \\ |P(f)|\exp\{j[\angle P(f)+\pi/2\}, & f<0 \end{cases}$$

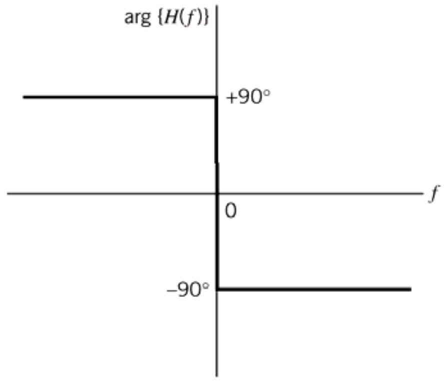
6.4 Hybrid amplitude/phase modulation schemes – Carrierless amplitude/phase modulation (CAP)

$$G_+(f)=2u(f)P(f)=P(f)+j\hat{P}(f)$$



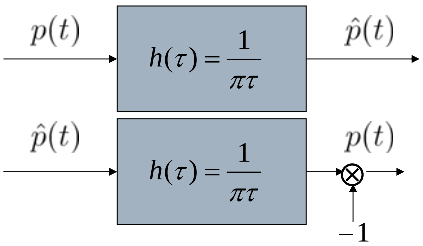
A2.3 Hilbert transform

- Hence, Hilbert transform is basically a *90 degree phase shifter*.



A2.3 Hilbert transform

Hilbert transform pair
$$\begin{cases} \hat{p}(t) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} p(\tau)\frac{1}{t-\tau}d\tau \\ p(t) = -\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} \hat{p}(\tau)\frac{1}{t-\tau}d\tau \end{cases}$$



A2.3 Hilbert Transform

- An important property of Hilbert transform is that:

$p(t)$ and $\hat{p}(t)$ are ortgogonal in the sense of integration. In other words, $\displaystyle\int_{-\infty}^{\infty} p(t)\hat{p}(t)dt = 0$. (See the proof in the next slide.)

The real and imaginary part of $g_+(t) = p(t) + j\hat{p}(t)$ are orthogonal to each other.

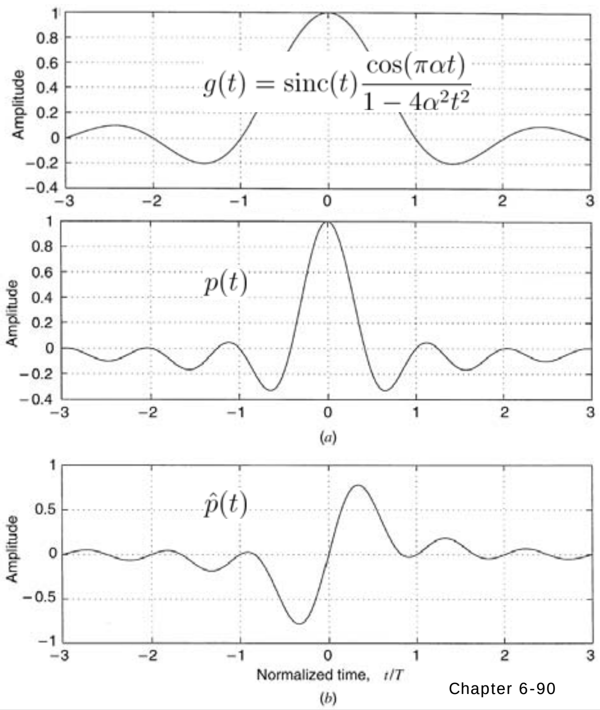
(*Examples of Hilbert transform pairs can be found in Table A6.4.*)

$$
\begin{aligned}
\boxed{\int_{-\infty}^{\infty} p(t)\hat{p}(t)dt} &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} P(f)e^{j2\pi ft}df\right)\hat{p}(t)dt \\
&= \int_{-\infty}^{\infty} P(f)\left(\int_{-\infty}^{\infty} \hat{p}(t)e^{j2\pi ft}dt\right)df \\
&= \int_{-\infty}^{\infty} P(f)\hat{P}(-f)df \\
&= \int_{-\infty}^{\infty} P(f)\left[-j\text{sgn}(-f)P(-f)\right]df \\
&= j\left(\int_{0}^{\infty} P(f)P(-f)df - \int_{-\infty}^{0} P(f)P(-f)df\right) \\
&= j\left(\int_{0}^{\infty} P(f)P(-f)df - \int_{0}^{\infty} P(-f)P(f)df\right) \\
&= 0, \text{ if } \int_{0}^{\infty} P(f)P(-f)df < \infty.
\end{aligned}
$$

6.4 Hybrid amplitude/phase modulation schemes – Carrierless amplitude/phase modulation (CAP)

- Properties of the passband in-phase and quadrature pulses in CAP
  - **Property 2:** $\hat{p}(t)$ is orthogonal to $p(t)$.
  - **Property 3:** $\hat{p}(t) * \lambda(t)$ is orthogonal to $p(t) * \lambda(t)$ for any linear filter $\lambda(t)$.
    - This is similar to use another pulse shaping function $g(t)*\lambda(t)$.
    - One is thus free to choose the pulse shaping function (to, e.g, improve the bandwidth efficiency) without affecting the orthogonality of two quadratures.

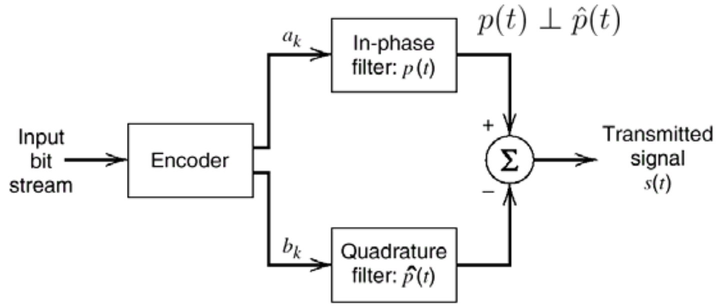
- Example 6.4: Bandwidth-efficient spectral shaping
  - α = 0.2



6.4 Hybrid amplitude/phase modulation schemes – Carrierless amplitude/phase modulation (CAP)

- Block diagram of CAP transmitter

$$p(t) \perp \hat{p}(t)$$

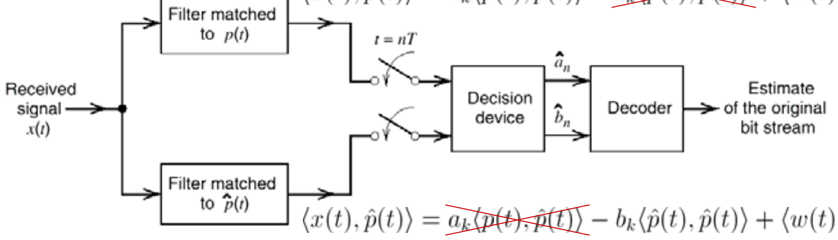


$$\boxed{s(t) = \mathrm{Re}\left\{ \sum_{k=-\infty}^{\infty} \tilde{A}_k \cdot g_+(t-kT) \right\}}$$

6.4 Hybrid amplitude/phase modulation schemes – Carrierless amplitude/phase modulation (CAP)

- Block diagram of CAP receiver

$$\langle x(t), p(t)\rangle = a_k\langle p(t), p(t)\rangle - \cancel{b_k\langle \hat{p}(t), p(t)\rangle} + \langle w(t), p(t)\rangle$$

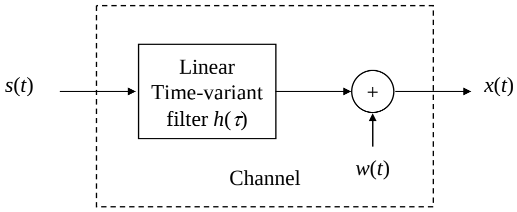


$$\langle x(t), \hat{p}(t)\rangle = \cancel{a_k\langle p(t), \hat{p}(t)\rangle} - b_k\langle \hat{p}(t), \hat{p}(t)\rangle + \langle w(t), \hat{p}(t)\rangle$$

$$\begin{aligned} x(t) &= s(t) + w(t) \\ &= a_k p(t) - b_k \hat{p}(t) + w(t), \text{ where } w(t) \text{ AWGN} \end{aligned}$$

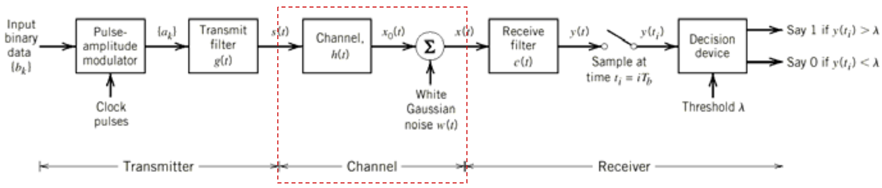
6.4 Hybrid amplitude/phase modulation schemes – Carrierless amplitude/phase modulation (CAP)

- How about channels with **intersymbol interferences**, in addition to AWGN?

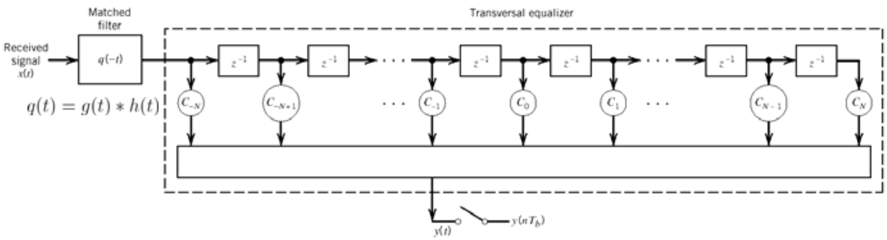


Recall the below terms:
1. Zero-forcing equalizer
2. Nyquist criterion/ISI
3. Noise enhancement
4. MMSE equalizer

- Recall in Section 4.9: Optimum linear receiver.

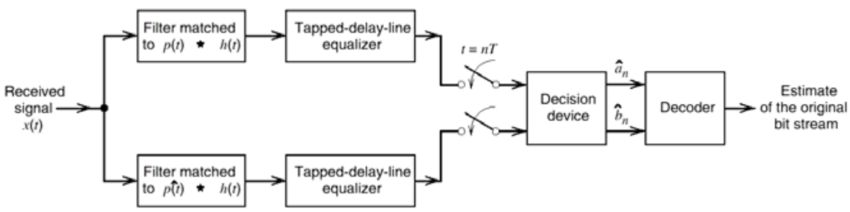


- Design of receiver $c(t)$ as an MMSE equalizer



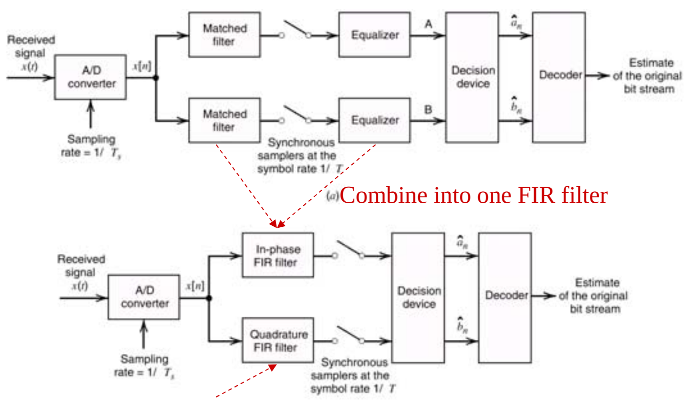
6.4 Hybrid amplitude/phase modulation schemes – Carrierless amplitude/phase modulation (CAP)

- This leads to Fig. 6.23 in text.



- Can we implement the above structure in “digital” form?

6.4 Hybrid amplitude/phase modulation schemes – Carrierless amplitude/phase modulation (CAP)



May be adaptive when required (b)

6.4 Hybrid amplitude/phase modulation schemes – Carrierless amplitude/phase modulation (CAP)

- Applications of CAP
  - Passband transmission of digital data over twisted-pair wiring of lengths less than 100m.
  - Data rates may range from 51 upto 155 Mbps with bandwidth being strictly limited to 30 MHz.

6.5 Coherent frequency-shift keying

- (M-ary) ASK, (M-ary) PSK and (M-ary) FSK are three major categories of digital modulations, in which QAM can be viewed/analyzed similarly to (M-ary) PSK.
- In this section, the last one, i.e., (M-ary) FSK, will be introduced and discussed.

6.5 Coherent frequency-shift keying

- Binary FSK

$$s_i(t) = \begin{cases} \sqrt{\dfrac{2E_b}{T_b}}\cos(2\pi f_i t), & 0 \le t < T_b \\ 0, & \text{elsewhere} \end{cases}$$

where $i = 1, 2$, f_i is a multiple of $1/T_b$,

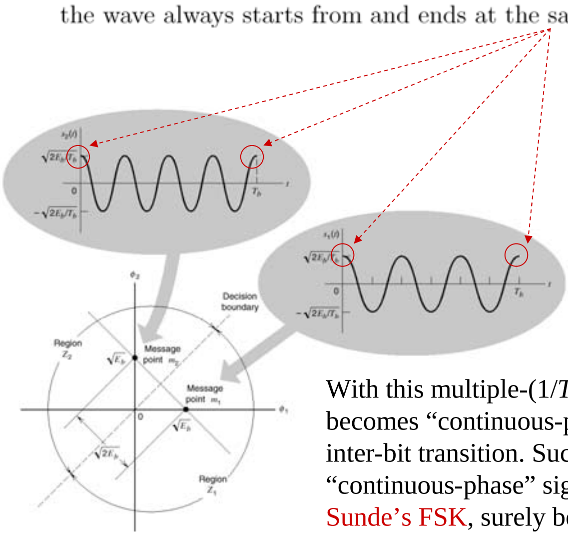
E_b is the transmitted energy per bit, and

T_b is the bit duration.

- Vector space analysis of binary FSK

$$\boldsymbol{s}_1 = \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix} \text{ and } \boldsymbol{s}_2 = \begin{bmatrix} 0 \\ \sqrt{E_b} \end{bmatrix} \text{ with } \begin{cases} \phi_1(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_1 t) \\ \phi_2(t) = \sqrt{\frac{2}{T_b}}\sin(2\pi f_2 t) \end{cases}$$

Since f_i is a multiple of $1/T_b$,
the wave always starts from and ends at the same point.



With this multiple-($1/T_b$) restriction, it becomes "continuous-phase" in every inter-bit transition. Such kind of forced "continuous-phase" signals, known as Sunde's FSK, surely belongs to the general continuous-phase FSK (CPFSK) family.